An overview of Modeling and Control of Flexible Robot

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1.0 Introduction

Robot manipulator consists of a series of link connected by joints to form a spatial mechanism. The joints are either revolute (rotary) or prismatic (telescope). In recent years, improvement in electric motor technology, coupled with new designs such as direct drive arms, have led to a rapid increase in speed and load carrying capabilities of manipulators. However, this has meant that the flexibility of the nominally rigid link has become increasingly significant.

Present generation manipulators are limited in their load carrying capacity by the requirement of rigidity. If the controller could compensate for link flexure, it would, in principle, possible to greatly increase the ratio of load to arm weight.

There are also other advantages arising from the use of flexible link:

- Lower energy consumption; lighter links have lower inertia and, therefore, require less power to produce the same acceleration as the rigid link with the same carrying capacity.
- Smaller actuators required; the reduced power requirement means that smaller and generally cheaper actuators can be used.
- Safer operation due to reduced inertia; in the event of collision. Less damage would be caused.
- Compliant structure; flexible links introduced mechanical compliance into robot structure. This is useful for delicate assembly operation; the link themselves being used for force/torque sensing.

Unfortunately, if the assumption of rigid link is relaxed, the equations describing the manipulator dynamic, become more complex. In addition to the gross motion of the arm, the deformational behavior of the individual links must be considered. The two effects interact further complicating the dynamics[1].

In this tutorial some of the techniques currently used for modeling and controlling flexible links robot will be presented. Two modeling method namely the assumed mode approach and the finite element method are presented in Section 2. Section 3 is concerned with control algorithms and we will conclude in section 4.

2.0 Modeling
2.1 Assume Mode Approach

2.1.1 Flexible Arm Modeling

Consider a uniform slender beam connected via a rigid hub to the armature of an electric motor; this beam also has a tip mass (payload) with inertia see Figure 2.1. the beam is assumed

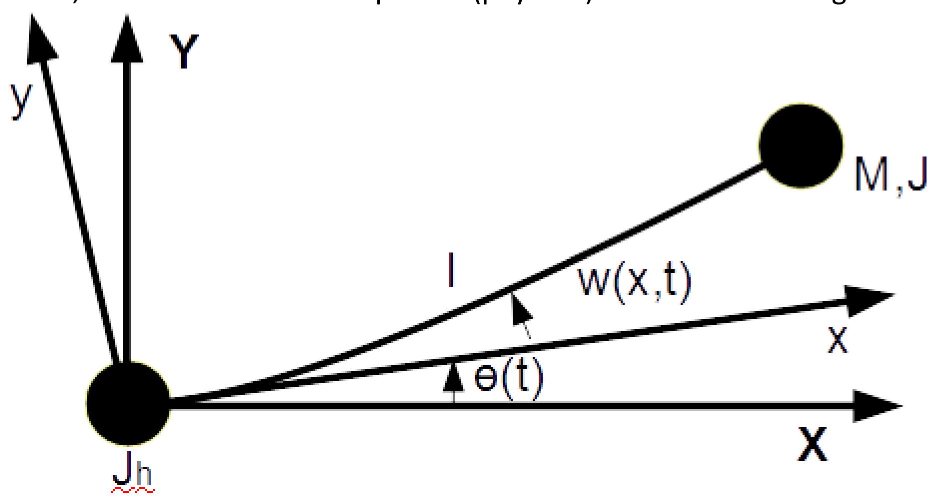

Figure 2.1: A Flexible link manipulator

To be initially straight, and to satisfy the assumption of Euler-Bernouilly beam theory. Note that x is the distance along the length of the beam; l is the length of the beam; δ is the thickness of the beam; m is the mass per unit length; EI is the flexural rigidity of the beam; r is the hub radius; J_h is the hub inertia; M is the payload mass; J is the payload inertia; J_b is the inertia of the beam about the motor armature.

The displacement of the joints along the deformed profile of the beam is described in terms of radial and circumferential coordinates and is related to the angles of the rotation of the hub θ and the flexural displacement of the beam as follows:

$$y(x,t) = w(x,t) + (x+r)\theta(t) \tag{2.1}$$

This displacement can be expanded using the assumed mode approach[1]. If we choose the cantilever mode, the displacement of any point on the beam is represented by the summation

$$y(x,t) = \sum \phi_i(x) q_i(t) + (x+r)\theta(t) \tag{2.2}$$

Where q_i is purely a function of time and includes an arbitrary multiplicative constant and ϕ_i, the mode shape, which is purely a function of the displacement along the beam and can be expressed as follows:

$\phi_i(x) = l(\cosh(\lambda_i x) - \cos(\lambda_i x)) - k_i(\sinh(\lambda_i x) - s\sin(\lambda_i x))$

Where $k_i = (\cos(\lambda_i l) + \cosh(\lambda_i l))/(\sin(\lambda_i l)\sinh(\lambda_i l))$ and λ_i are root of

$1 + \cos(\lambda_i l)\cosh(\lambda_i l) = 0$

Using this displacement, the kinetic and potential energy are given as [1]

$T_k = \frac{1}{2}\left\{(J_h + J_b)\dot{\theta}^2 + 4ml^3\dot{\theta}\sum_{i=1}^{\infty}\dot{q}/\lambda_i^2 + ml^3\sum_{i=1}^{\infty}\dot{q}_i^2\right\}$ and $V_e = ml^3\sum_{i=1}^{\infty}\omega_i^2\dot{q}_i^2$

Define the Lagrangian to be:

$$L = T_k - V_e$$

the dynamic equation is given as:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \tau \qquad (2.3)$$

for $i = 0,1,2.....$ and τ the motor torque. Equation (2.3) yields an infinite set of couple second-order differential equations. Retaining the first n elastic modes and writing these equations in matrix form, we obtain

$$M_c\ddot{q} + K_c q = Q \qquad (2.4)$$

Where

$$M_c = \begin{bmatrix} J_T & 2\frac{ml^3}{\lambda_1^2} & .. & 2\frac{ml^3}{\lambda_n^2} \\ 2\frac{ml^3}{\lambda_1^2} & ml^3 & . & 0 \\ . & . & . & . \\ 2\frac{ml^3}{\lambda_n^2} & 0 & . & ml^3 \end{bmatrix} \qquad K_c = \begin{bmatrix} 0 & 0 & . & 0 \\ . & ml^3\omega_1^2 & . & 0 \\ . & . & . & . \\ 0 & 0 & .. & ml^3\omega_n^2 \end{bmatrix} \qquad q = \begin{bmatrix}\theta & q_1 & .. & q_n\end{bmatrix}^T$$

$Q = \begin{bmatrix}\tau & 0 & .. & 0\end{bmatrix}$

(2.5)

Equation (2.5) can be transformed into the following state-space form[1]:

$$\dot{x} = Ax + B\tau \qquad (2.5)$$

Where

$$A = \begin{bmatrix} 0 & -M_c^{-1}K_c \\ I & 0 \end{bmatrix} \qquad x = \begin{bmatrix}\dot{\theta}_0 & \dot{q}_1 & .. & \dot{q}_n & \theta_0 & q_1 & .. & q_n\end{bmatrix}^T$$

$$B = \frac{1}{J_T}\begin{bmatrix}1 & \phi_1^{'} & .. & \phi_n^{'} & 0 & 0 & 0 & 0\end{bmatrix}^T$$

The output equation is given as $z = Cx$ in which C depends on the choice of the desired outputs. For the three outputs: hub angle $(\theta_h(t) = y^{'}(0,t))$, tip position $(y_t(t) = y(l,t))$ and root strain $(\sigma(t) = (\delta/2)y^{''}(0,t),$ take the form:

$$C = \begin{bmatrix} 0 & . & . & 1 & \phi_1^{'}(0) & . & . & \phi_n^{'}(0) \\ 0 & . & . & l & \phi_1^{'}(l) & . & . & \phi_n^{'}(l) \\ 0 & . & 0 & 0 & \frac{\delta}{2}\phi_1^{''}(0) & . & . & \frac{\delta}{2}\phi_n^{''}(0) \end{bmatrix} \quad (2.6)$$

With $\phi^{'}$ and $\phi^{''}$ are the first and second derivative.

2.1.2 Two- Flexible Arm Modeling

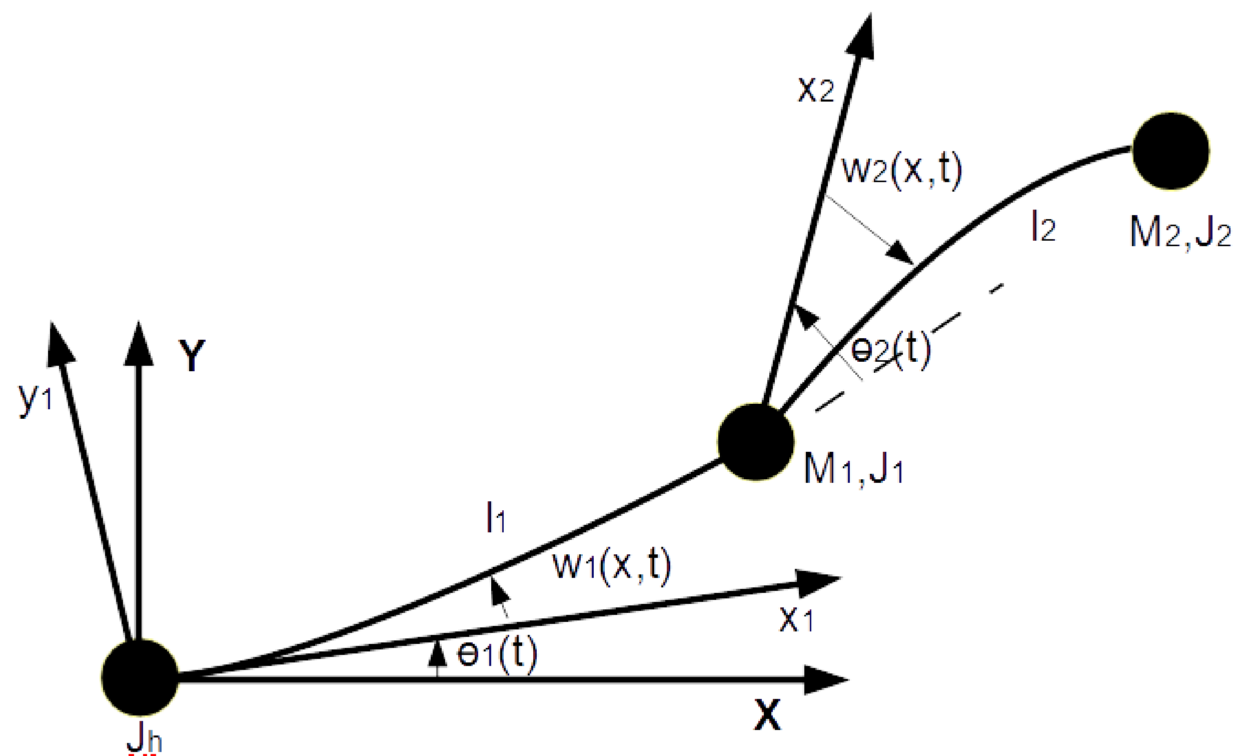

Figure 2.2: A two-flexible link manipulator

Consider the two-link manipulator presented on Figure 2.2. Assume that $link_1$ has a cross sectional area $A_1(x_1)$, and a length l_1, with flexible mode represented by $q_{11}, q_{12}, ... q_{1m1}$ similarly that $link_2$ has cross sectional area $A_2(x_2)$, with flexible mode $q_{21}, q_{22}, ... q_{2m2}$ where $m_1 + m_2 = m$ the total number of flexible mode, and $X - Y$ the stationary world frame and θ_1 is the angle of the tangent at $x = 0$ and with respect to $X - axis$ and θ_2 is the angle that $x_2 - axis$ makes relative to the slope of the end point of $link_1$. The mode shape functions are assumed to be clamped at the actuator end (clamped-mass or clamped-free).

Let us denote the description of a point on $link_1$ written in the $X - Y$ plane by[2]

$$r_1(x_1,t) = T(\theta_1)\begin{bmatrix} x_1 \\ \sum_{i=1}^{m_1}\phi_{1i}(x_1)q_{1i}(t) \end{bmatrix} \tag{2.7}$$

Where $T(\theta_1)$ is the rotation matrix of the $x_1 - y_1$ frame, i.e,

$$T(\theta_1) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \tag{2.8}$$

Therefore the kinetic energy due to $link_1$ is given by

$$T_{k1} = \frac{1}{2}\int_0^{l_1} \rho A_1(x_1)\dot{r}_1^T \dot{r}_1 dx_1 \tag{2.9}$$

Similarly the description of a point corresponding to x_2 on the second link is given in the $X - Y$ frame by

$$r_2(x_2,t) = T(\theta_1)(\begin{bmatrix} l_1 \\ y_1(l_1) \end{bmatrix} + T_2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) \tag{2.10}$$

Where

$$y_1(l_1) = \sum_{i=1}^{m_1}\phi_{1i}(l_1)q_{1i}(t) \tag{2.11}$$

$$T_2 = \begin{bmatrix} \cos(\theta_2 + \sum_{i=1}^{m_1}\phi_{1il}^{'}q_{1i}) & -\sin(\theta_2 + \sum_{i=1}^{m_1}\phi_{1il}^{'}q_{1i}) \\ \sin(\theta_2 + \sum_{i=1}^{m_1}\phi_{1il}^{'}q_{1i}) & \cos(\theta_2 + \sum_{i=1}^{m_1}\phi_{1il}^{'}q_{1i}) \end{bmatrix} \tag{2.12}$$

$$\phi_{1il}^{'} = \frac{d}{dx_1}(\phi_{1i}(x_1))|_{x_1=l_1} \tag{2.13}$$

Hence the kinetic energy due to the second link is given as follows:

$$T_{k2} = \frac{1}{2}\int_0^{l_2} \rho A_2(x_1)\dot{r}_2^T \dot{r}_2 dx_2 \tag{2.14}$$

The hub kinetic energy is given by

$$T_h = \frac{1}{2} J_h \dot{\theta}_1^2 \tag{2.15}$$

The kinetic energy due to M_1 and J_1 which denote the mass and moment of inertia of the case and stator of the second motor is given by

$$T_{kmotor} = \frac{1}{2} M_2 \dot{r}_2^T \dot{r}_2 \,|_{x_2 = l_2} + \frac{1}{2} J_1 (\dot{\theta}_1 + \sum_{i=1}^{m_1} \phi'_{1il} \dot{q}_{1i})^2 \tag{2.16}$$

And finally the payload kinetic energy is

$$T_{kp} = \frac{1}{2} M_2 \dot{r}_2^T \dot{r}_2 \,|_{x_2 = l_2} + \frac{1}{2} J_2 (\dot{\theta}_1 + \dot{\theta}_2 + \sum_{i=1}^{m_1} \phi'_{1il} \dot{q}_{1i} + \sum_{i=1}^{m_2} \phi'_{2il} \dot{q}_{2i})^2 \tag{2.17}$$

The total kinetic energy is given as follows

$$T_k = T_{k1} + T_{k2} + T_h + T_{kmot} + T_{kp} \tag{2.18}$$

The elastic potential energy is obtained from

$$V_e = \frac{1}{2} \sum_{i=1}^{m_1} \sum_{l=1}^{m_1} q_{1k} q_{1l} K_{ikl} + \frac{1}{2} \sum_{i=1}^{m_2} \sum_{l=1}^{m_2} q_{2k} q_{2l} K_{ikl} \tag{2.19}$$

Which can be written in matrix form

$$V_e = \frac{1}{2} q^T K q \tag{2.20}$$

And

$$\begin{aligned} K_{1kl} &= \int_0^{l_1} EI_{z1}(x_1) \frac{d^2 \phi_{1k}(x_1)}{dx_1^2} \frac{d^2 \phi_{1k}(x_1)}{dx_1^2} dx_1 \\ K_{2kl} &= \int_0^{l_1} EI_{z2}(x_2) \frac{d^2 \phi_{2k}(x_2)}{dx_2^2} \frac{d^2 \phi_{1k}(x_2)}{dx_2^2} dx_2 \end{aligned} \tag{2.21}$$

E is the modulus of elasticity of the material and $I_{z1}(x_1)$, $I_{z2}(x_2)$ are the area moments of inertia about the axis of rotation z_1 and z_2 at x_1 and x_2.

Note that for a uniform manipulator the cross product terms are zero if orthogonal shape function (e.q clamp free) are used.

If gravity is present, the potential energy due to gravity can also be added to V_e. The dynamic equation is derive in the absence of gravity. Denoting the degree of freedom by:

$$z^T = \begin{bmatrix}\theta_1 & \theta_2 & q_{11} & q_{12} & .. & q_{1m_1} & q_{21} & q_{21} & .. & q_{2m_2}\end{bmatrix}^T = \begin{bmatrix}\theta^T & q^T\end{bmatrix}$$

The Lagrangian equation for the system is given by

$$L = T_k - V_z = \frac{1}{2}(\dot{z}^T M\dot{z} - z^T Kz) \qquad (2.21)$$

From which the system dynamic can be obtained using the Euler Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \tau_g \qquad (2.22)$$

Where τ_g is the generalized vector of torques which for the case of clamped mode shape is given by

$$\tau_g = \begin{bmatrix} I_{2\times2} \\ O_{2\times2} \end{bmatrix}\tau = Q\tau \qquad (2.23)$$

Where τ is the actuator torque at the joint, $O_{2\times2}$ is a 2×2 matrix of zero elements. It follows that the dynamic equation can be written as follows:

$$M(\theta, q)\ddot{z} + V(z,\dot{z}) + K_a z = Q\tau \qquad (2.24)$$

Where the ith element of $V(z,\dot{z})$ is

$$V_i(z,\dot{z}) = e_i^T \sum_{j=1}^{m+2} \dot{z}_j \frac{\partial M}{\partial z_j} z - \frac{1}{2}\dot{z}^T \frac{\partial M}{\partial z_i}\dot{z}, \qquad i = 1...., m+2 \qquad (2.25)$$

With e_i the unity vector, $V(z,\dot{z})$ containing the centrifugal and Coriollis terms and K_a is given by

$$K_a = \begin{bmatrix} O_{2\times2} & O_{2\times m} \\ O_{m\times2} & K_{m\times m} \end{bmatrix} \qquad (2.26)$$

$O_{2\times m}$, is a $2\times m$ matrix of zero elements and $O_{m\times2}$, is a matrix of $m\times2$ zero elements and $K_{m\times m}$ is an $m\times m$ matrix of Kij a generic stiffness constant elements.

2.2 Finite Element method

The overall finite element approach involves treating each link of the manipulator (say $link_i$) as an assemblage of n_i elements of length l_i. For each of these elements (say ij where subscript ij refers to the jth element of $link_i$) the kinetic energy T_{ij} and potential energy V_{ij} are computed in terms of a suitably selected system of n generalized variables that are used in order to compute the Lagrangian L which in turn is used to derive the dynamic equations. The procedure can be described as follows[3]:

1. Let $L=1$ and $i=1$
2. Divide $link_i$ into n_i elements of length l_i
3. Compute T_{ij} and V_{ij} for the generic element ij
4. Compute the link kinetic and potential energies by combining the finite element energies as follows: $T_i=\sum_{j=1}^{n_i} T_{ij}$ and $V_i=\sum_{j=1}^{n_i} V_{ij}$
5. Compute the Link Lagrangian as follows: $L=L+L_i$
6. $i=i+1$
7. Is $i>m$ where m is the number of link If no got to 2 if yes go to 8
8. Use the Lagrangian to the derive the dynamic equations as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=\tau \quad i=1,....m \qquad (2.27)$$

9. Stop

2.2.1 Kinetic Description

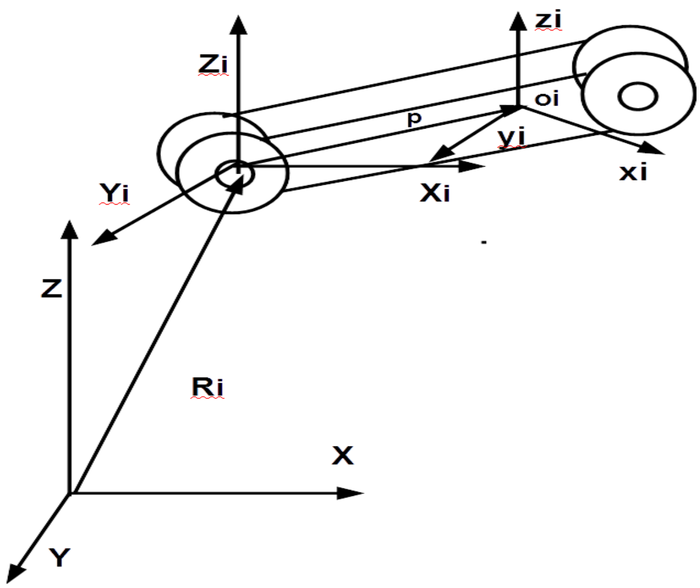

Figure 2.3: Representation of an arbitrary point on a rigid link

The general displacements of a point is described using Chasles theorem[6]. It defines an arbitrary displacement as a sum of the translation of a point and a rotation along the axis of rotation. According to Chasles theorem , the arbitrary displacement of a point can be defined as

$$r=R+Hu \qquad (2.28)$$

Where $r=\begin{bmatrix} r_x & r_y & r_z \end{bmatrix}^T$ is the global position vector of an arbitrary point, $R=\begin{bmatrix} R_x & R_y & R_z \end{bmatrix}^T$ is the position vector of the body coordinate system. H is the coordinate transformation matrix, and $u = \lfloor u_x \quad u_y \quad u_z \rfloor$ is the local position vector defined with respect to the body coordinate system. The transformation matrix H is defined as follows:

$$H = \begin{bmatrix} 2(\beta_0^2 + \beta_1^2) - 10 & 2(\beta_1\beta_2 - \beta_0\beta_2) & 2(\beta_1\beta_3 + \beta_0\beta_2) \\ 2(\beta_1\beta_2 + \beta_0\beta_2) & 2(\beta_0^2 + \beta_2^2) - 1 & 2(\beta_2\beta_3 - \beta_0\beta_1) \\ 2(\beta_1\beta_3 - \beta_0\beta_2) & 2(\beta_2\beta_3 + \beta_0\beta_1) & 2(\beta_0^2 + \beta_3^2) - 1 \end{bmatrix} \tag{2.29}$$

where $\beta_0, \beta_1, \beta_2, \beta_3,$ are the Euler parameters. These quantities are defined as

$$\begin{aligned} \beta_0 &= \cos(\frac{\beta}{2}) \\ \beta_1 &= v_1 \sin(\frac{\beta}{2}) \\ \beta_2 &= v_2 \sin(\frac{\beta}{2}) \\ \beta_3 &= v_3 \sin(\frac{\beta}{2}) \end{aligned} \tag{2.30}$$

in which v_1, v_2 and v_3 are components of the unit vector v along the axis of rotation. v is the angle of rotation.

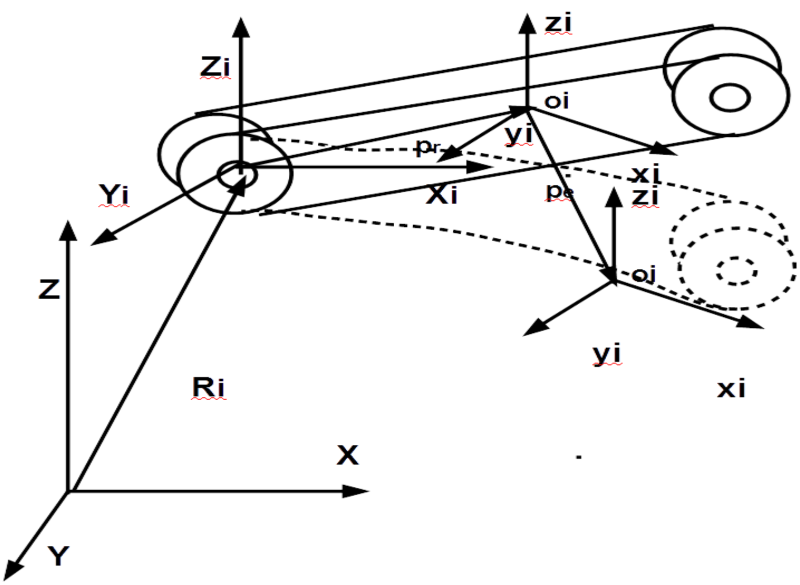

Figure 2.4: Representation of an arbitrary point on a flexible link

2.2.2 Kinetics of Flexible link

The kinetic equations that can define an arbitrary displacement of a flexible link i is derived using floating reference frame formulation [6]. Floating reference frame formulation uses two sets of coordinates i.e., body reference coordinates and elastic coordinates. The body reference coordinates describe the position and orientation of body coordinate system X_i, Y_i, Z_i with respect to the global coordinate system X, Y, Z. The elastic coordinates describe the local displacements of flexible $link_i$ with respect to the body coordinate system X_i, Y_i, Z_i.

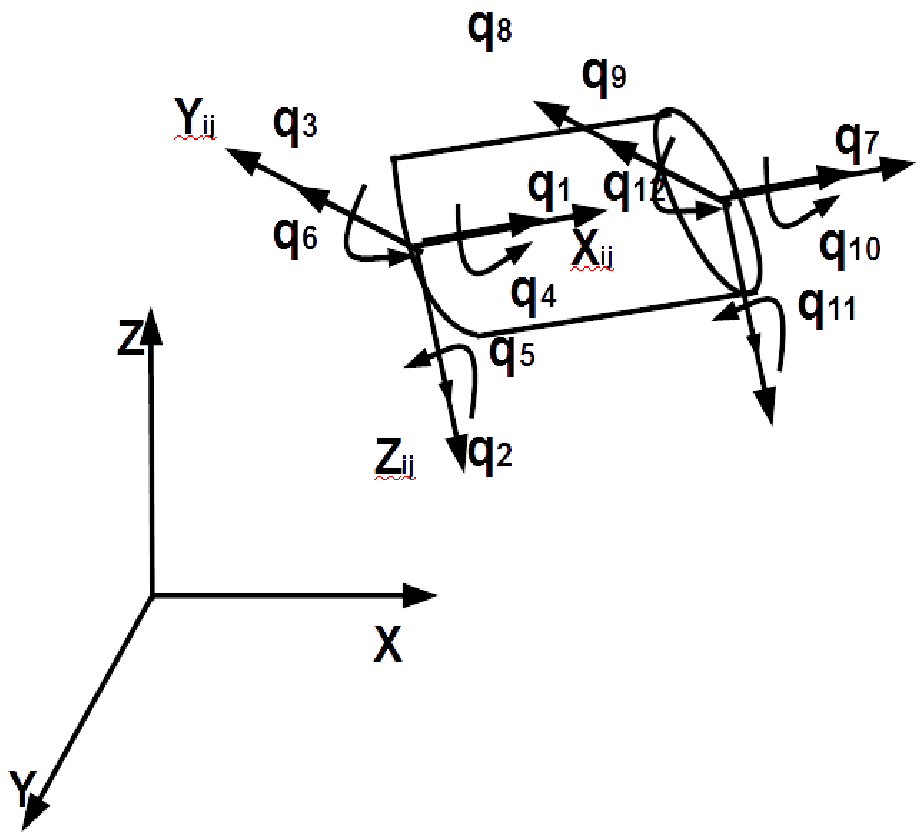

Figure 2.5: Elastic coordinate on finite element

The elastic deformations of flexible link are approximated using the finite element method to obtain finite set of elastic coordinates. The elastic coordinates of finite element shown in Figure 2.5 is defined using element coordinate system X_{ij}, Y_{ij}, Z_{ij} with respect to body coordinate system X_i, Y_i, Z_i. The position vector of an arbitrary point on flexible $link_i$ is defined as

$$r_i = R_i + H_i u_i \qquad (2.31)$$

where $R_i = \begin{bmatrix} R_x & R_y & R_z \end{bmatrix}^T$ is the position vector of the body coordinate system X_i, Y_i, Z_i. H_i is the transformation matrix defined using equation [6],and u_i is the local position vector defined with respect to X_i, Y_i, Z_i. For the flexible link, the local position vector u_i is defined as the sum of undeformed position vector and elastic deformation vector. The local position vector u_i is written as

$$u_i = u_i^r + u_i^e \qquad (2.32)$$

where u_i^r is the undeformed position vector, and u_i^e is elastic deformation vector that is defined as

$$u_i = S_i q_i^e \tag{2.33}$$

in which S_i is the shape function matrix, and q_i^e is the elastic coordinates vector[6]. The shape function S_i is defined as

$$S_i = \begin{bmatrix} 1-\xi & 0 & 0 \\ 6(\xi-\xi^2)\eta & 1-3\xi^2+2\xi^3 & 0 \\ 6(\xi-\xi^2)\xi & 0 & 1-3\xi^2+2\xi^3 \\ 0 & -(1-\xi)l\zeta & -(1-\xi)l\eta \\ (1-4\xi+3\xi^2)l\zeta & 0 & (-\xi+2\xi^2-\xi^3)l \\ (-1+4\xi-3\xi^2)l\eta & (\xi-2\xi^2+\xi^3)l & 0 \\ \xi & 0 & 0 \\ 6(-\xi+\xi^2)\eta & 3\xi^2-2\xi^3 & 0 \\ 6(-\xi+\xi^2)\zeta & 0 & 3\xi^2-2\xi^3 \\ 0 & -l\xi\zeta & -l\xi\eta \\ (-2\xi+3\xi^2)l\zeta & 0 & (\xi^2-\xi^3)l \\ (2\xi-3\xi^2)l\eta & -(\xi^2-\xi^3)l & 0 \end{bmatrix} \tag{2.34}$$

With $\xi = \frac{u_x}{l}; \eta = \frac{u_y}{l}; \zeta = \frac{u_z}{l}$ and l is length of element, and u_x, u_y, u_z are spatial coordinates along element axis.

The velocity vector of an arbitrary point on the flexible $link_i$ is given as follows:

$$\dot{r}_i = \dot{R}_i + H_i(\omega_i \times u_i) + H_i S_i q_i^e \tag{2.35}$$

where ω_i is angular velocity vector defined in body coordinate system X_i, Y_i, Z_i. It is expressed as

$$\omega_i = G_i \dot{\beta}_i \tag{2.36}$$

$$G_i = 2\begin{bmatrix} -\beta_0 & \beta_0 & \beta_3 & -\beta_2 \\ -\beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ -\beta_3 & \beta_2 & -\beta_1 & \beta_0 \end{bmatrix} \tag{2.37}$$

Equation [2.35] can be written as follows

$$\dot{r}_i = L_i \dot{q}_i \tag{2.38}$$

Where $L_i = \begin{bmatrix} I & H_i \tilde{u}_i G_i & H_i S_i \end{bmatrix}$ and $\dot{q}_i$ is generalized velocities of flexible $link_i$ defined in absolute coordinate $\dot{q}_i = \begin{bmatrix} \dot{R}_i & \dot{\beta}_i & \dot{q}_i^e \end{bmatrix}^T$ system.

The acceleration vector of an arbitrary point on the flexible link i is obtained by differentiating equation [2.38]. It is written as

$$\ddot{r}_i = L_i \ddot{q}_i + H_i (\tilde{\omega}_i)^2 u_i + 2H_i S_i q_i^e \qquad (2.39)$$

With $\ddot{q}_i = \begin{bmatrix} \ddot{R}_i & \ddot{\beta}_i & \ddot{q}_i^e \end{bmatrix}^T$ in which $\ddot{q}_i$ is generalized accelerations of the flexible $link_i$ defined in absolute coordinate system.

2.2.1 Dynamic System Modeling

The dynamics equations of motion are derived using the principle of virtual work in absolute coordinate system[6]. The equations of motion can be rearranged as follows

$$M_{ij} \ddot{q}_{ij} = Q_{ij}^e + Q_{ij}^v + Q_{ij}^s \qquad (2.40)$$

where Q_{ij}^e are the applied external forces. Q_{ij}^v and Q_{ij}^s are respectively the quadratic velocity term and elastic forces acting on element ij.

$$M_{ij} = \int_{V_{ij}} \rho_{ij} \begin{bmatrix} I & -H_i \tilde{u}_{ij} G_i & H_i S_{ij} \\ -H_i \tilde{u}_{ij} G_i & G_i^T \tilde{u}_{ij}^T u_{ij} G_i & G_i^T \tilde{u}_{ij}^T S_{ij} \\ H_i S_{ij} & G_i^T \tilde{u}_{ij}^T S_{ij} & S_{ij}^T S_{ij} \end{bmatrix} dV_{ij}, \; Q_{ij}^v = \int_{V_{ij}} \rho_{ij} \begin{bmatrix} I \\ G_i^T \tilde{u}_{ij}^T H_i^T \\ S_{ij}^T H_{ij}^T \end{bmatrix} dV_{ij} \qquad (2.41)$$

$Q_{ij}^s = -q_{ij}^{eT} K_{ij}^e$ where K_{ij}^e is the element stiffness matrix defined as $K_{ij}^e = \int_{V_{ij}} (D_{ij} S_{ij})^T E_{ij} D_{ij} S_{ij} dV_{ij}$

with D_{ij} is the differential operator, S_{ij} is the element shape function matrix and E_{ij} is the elastic coefficient.

The equations of motion of the flexible $link_i$ is defined as

$$M_i \ddot{q}_i = Q_i^e + Q_i^v + Q_i^s \qquad (2.42)$$

With $M_i = \sum_{j=1}^{m} M_{ij}, Q_i^e = \sum_{j=1}^{m} Q_{ij}^e, Q_i^v = \sum_{j=1}^{m} Q_{ij}^v, Q_i^s = \sum_{j=1}^{m} Q_{ij}^s$. m number of finite elements per link.

The equations of motion in absolute coordinate system for an n links manipulator is given as follows[6]

$$M\ddot{q} = Q^e + Q^v + Q^s \qquad (2.42)$$

$$M = \begin{bmatrix} M_1 & 0 & . & . & . & 0 \\ 0 & M_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & M_n \end{bmatrix}, \ddot{q} = \begin{bmatrix} \ddot{q}_1 & \ddot{q}_2 & \dots & q_n \end{bmatrix}, Q^e = \begin{bmatrix} Q_1^e & Q_2^e & \dots & Q_n^e \end{bmatrix}^T \qquad (2.43)$$

$$Q^v = \begin{bmatrix} Q_1^v & Q_2^v & \dots & Q_n^v \end{bmatrix}^T, \quad Q^s = \begin{bmatrix} Q_1^s & Q_2^s & \dots & Q_n^s \end{bmatrix}^T \qquad (2.44)$$

Where Q_i^e are the external forces applied on flexible $link_i$. Q_i^v and Q_i^s are respectively the quadratic velocity term and elastic forces acting on flexible $link_i$.

3.0 Control

Motion control of robot manipulator is important in order to achieve high speed operations. In this section few model based tracking algorithm for Flexible link manipulator will be presented. The object of the control is to follow the desired trajectory in addition to minimizing vibrations of the end-effector.

The design of controller for flexible link manipulators is difficult due to its under actuation and non-minimal phase nature of the dynamic. Under actuation is caused by the fact that a finite number of actuators is used to control infinite degrees of freedom that arise due to link flexibility. Non-minimum phase nature occurs because of non-collocation of actuators and sensors.

The equations that describe the dynamic of flexible link manipulator can be written as follows:

$$M(q)\ddot{q} + V(q,\dot{q})\dot{q} + D\dot{q} + Kq = B\tau \tag{3.1}$$

where $q = [q_r \quad q_f]^T$ are rigid and elastic coordinates of the manipulator; q_r is the $n \times 1$ vector that represents rigid body rotations of the n manipulator joints, and q_f is the $m \times 1$ vector that represent elastic coordinates of the link. The number of elastic coordinates depends on the number of finite elements used to discretize the link or the number of flexible modes that remain after truncation of the model, when using the assumed mode method. $M(q)$ is the Inertia matrix, $V(q,\dot{q})\dot{q}$ is the Coriolis and centrifugal vector, $D\dot{q}$ is the frictional and damping forces, Kq represents the internal forces due to body elasticity. Input matrix B maps the external torque into generalized forces of the system.

The equations of motion explicitly written in rigid and elastic coordinates are given as follows:

$$\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} \ddot{q}_r \\ \ddot{q}_f \end{bmatrix} + \begin{bmatrix} V_{rr} & V_{rf} \\ V_{fr} & V_{ff} \end{bmatrix}\begin{bmatrix} \dot{q}_r \\ \dot{q}_f \end{bmatrix} + \begin{bmatrix} D_{rr} & 0 \\ 0 & D_{ff} \end{bmatrix}\begin{bmatrix} \dot{q}_r \\ \dot{q}_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_{ff} \end{bmatrix}\begin{bmatrix} q_r \\ q_f \end{bmatrix} = \begin{bmatrix} B_r \\ B_f \end{bmatrix}\tau \tag{3.2}$$

The actuators are assumed to be placed at manipulator joints. Thus, the input matrix B is expressed as $B_r = I_{n\times n}$ and $B_f = 0_{m\times m}$. The model inversion of equation [6], that maps input torque and desired output trajectory, depends on the rigid and elastic coordinates of the system. If the desired output trajectory is the tip trajectory, the system is unstable due to non-minimum phase nature.

3.1 One flexible Link $H\infty$ Controller Design

This section present a controller designed using $H\infty$ design techniques. $H\infty$ is a norm that is defined as follows

$\|F\|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}[F(s)] = \sup_{\omega\in\Re} \bar{\sigma}[F(j\omega)]$ in the case of a transfer function, it can be view as the peak on the Bode magnitude plot of $|F(j\omega)|$. We are interested in designing a controller that minimize this peak.

The model of a single flexible link robot manipulator is linear. The state space representation is given as follows:

$$\begin{aligned} \dot{x} &= Ax + B\tau \\ y &= Cx \end{aligned} \tag{3.3}$$

where A, B, C are given by equations (2.5), (2.6). The transfer matrix is given as follows:

$$P(s) = C(SI - A)^{-1} B \tag{3.4}$$

Defined the multiplicative unstructured uncertainties as follows:

$$\Delta_m(s) = W_2 \Delta W_1 = P_0^{-1}(s)(P(s) - P_0(s)) \tag{3.5}$$

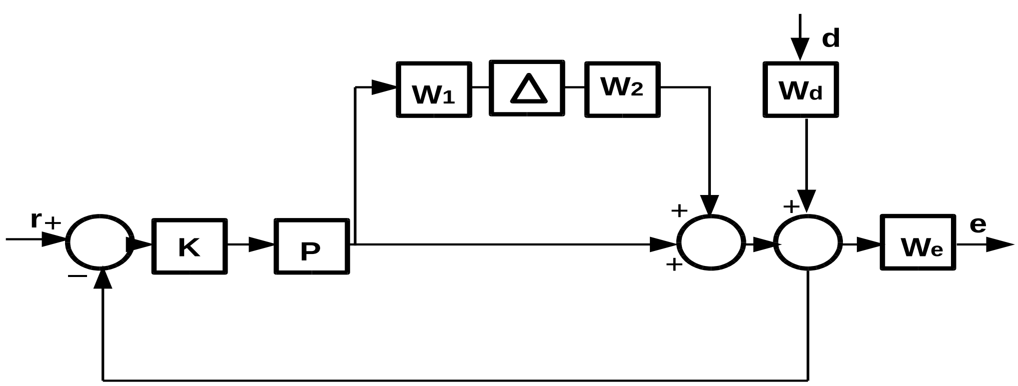

Figure 3.1 $H\infty$ controller design

Where $\|\Delta\|_\infty \le 1$ and $P_0(s)$ is the nominal transfer matrix defined for the nominal value of the hub and the load inertia J_{h0}, J_{t0} as follows $P_0(s) = P(s; J_{h0}, J_{t0})$, d is a disturbance, e output error.

We are interested in synthesising a controller $K(s)$ that minimize the $H\infty$ mixed sensitivity norm defined as follows:

$$\begin{Vmatrix} W_e S_o W_d \\ W_1 T_o W_2 \end{Vmatrix}_\infty \tag{3.6}$$

Where $L_o = PK$ is the output loop transfer matrix, $S_o = (I + L_o)^{-1}$ is the output sensitivity matrix, $T_o = I - S_o = L_o(I + L_o)^{-1}$ is the output complementary sensitivity matrices all of them defined in the nominal condition and W_1, W_2, W_e, W_d, are frequency-dependent scaling matrices[12]. The solution of this problem is a dynamic compensator whose transfer matrix is given by $K(s)$.

3.2 Lyapunov Based Nonlinear Controller

Consider the dynamic equation (2.46). Lyapunov method will be used in order to derive the control law. Let's define the position error along the trajectory as follows:

$e = \begin{bmatrix} q_{rd} - q_r \\ q_{fd} - q_f \end{bmatrix}$ where q_{rd} and q_{fd} are the desired rigid and flexible coordinates. q_{fd} is set to zero to suppress vibrations. Therefore $e = \begin{bmatrix} q_{rd} - q_r \\ -q_f \end{bmatrix}$. Defined the sliding surface as follows:

$S = \dot{e} + \lambda e = \begin{bmatrix} s_r = \dot{e}_r + \lambda_r e_r \\ s_f = \dot{e}_f + \lambda_f e_f \end{bmatrix}$ where $\lambda = \begin{bmatrix} \lambda_r & 0 \\ 0 & \lambda_f \end{bmatrix}$

The error dynamics of the system in term of s_r, s_f is given as follows.

$$\begin{bmatrix} M_{rr} & M_{rf} \\ M_{fr} & M_{ff} \end{bmatrix}\begin{bmatrix} \dot{s}_r \\ \dot{s}_f \end{bmatrix} + \begin{bmatrix} V_{rr} & V_{rf} \\ V_{fr} & V_{ff} \end{bmatrix}\begin{bmatrix} s_r \\ s_f \end{bmatrix} + \begin{bmatrix} D_{rr} & 0 \\ 0 & D_{ff} \end{bmatrix}\begin{bmatrix} s_r \\ s_f \end{bmatrix} + \begin{bmatrix} K_{vr} & 0 \\ 0 & K_{vf} \end{bmatrix}\begin{bmatrix} s_r \\ s_f \end{bmatrix} = \begin{bmatrix} \tau_m + K_{vr} s_r - \tau \\ \tau_a \end{bmatrix}$$
(2.47)

Where

$$\tau_m = M_{rr}(\ddot{q}_{rd} + \lambda_r \dot{e}_r) + M_{rf}(-\lambda_f \dot{e}_f) + V_{rr}(\dot{q}_{rd} + \lambda_r e_r) + V_{rf}(-\lambda_f e_f) + D_{rr}(\dot{q}_{rd} + \lambda_r e_r) \tag{3.7}$$

$$\tau_a = M_{fr}(\ddot{q}_{rd} + \lambda_r \dot{e}_r) + M_{ff}(-\lambda_f \dot{e}_f) + V_{fr}(\dot{q}_{rd} + \lambda_r e_r) + V_{ff}(-\lambda_f e_f) + D_{ff}(-\lambda_f e_f) + K_{ff} e_f + K_{vf} s_f$$
(2.49)

It can be shown, by using Lyapunov method, that the following control law[6]:

$$\tau = \tau_m + K_{vr} s_r + \tau_f \tag{3.8}$$

With $\tau_f = \dfrac{(1+k)s_r}{\|s_r\|^2 + \varepsilon}(s_f^T \tau_a)$ (2.51) and k solution of $\dot{k} = \dfrac{1}{k}\left(\dfrac{k(\|s_r\|^2 - \varepsilon}{\|s_r\|^2 + \varepsilon}\right)(s_f^T \tau_a)$ (2.52)

Will stabilized the Flexible link robot system . The architecture of this controller is given in Figure [3.2]

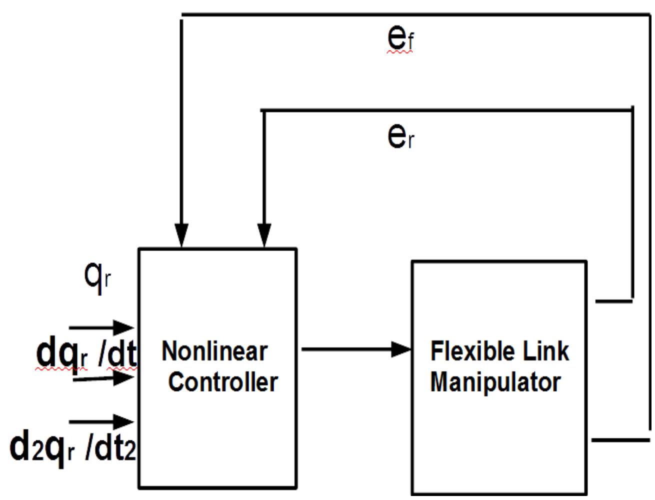

Figure 3.2 A nonlinear control architecture.

3.3 Adaptive controller

The dynamics of the flexible manipulator[2.46] can be is expressed in terms of linear type parametric model as follows:

$$Z_1\Theta_1 = M_{rr}(\ddot{q}_{rd} + \lambda_r \dot{e}_r) + M_{rf}(-\lambda_f \dot{e}_f) + V_{rr}(\dot{q}_{rd} + \lambda_r e_r) + V_{rf}(-\lambda_f e_f) + D_{rr}(\dot{q}_{rd} + \lambda_r e_r) \quad (2.53)$$

$$Z_2\Theta_2 = M_{fr}(\ddot{q}_{rd} + \lambda_r \dot{e}_r) + M_{ff}(-\lambda_f \dot{e}_f) + V_{fr}(\dot{q}_{rd} + \lambda_r e_r) + V_{ff}(-\lambda_f e_f) + D_{ff}(-\lambda_f e_f) + K_{ff} e_f + K_{vf} s_f$$
(2.54)

where Z_1 and Z_2 are $n \times r_1$, $m \times r_2$ regression matrix for appropriate $r_1, r_2 > 0$; and θ_1, θ_2 are unknown constant parameters.

It can be shown, by using Lyapunov method, that the following control law[6]:

$$\tau = Z_1 \dot{\hat{\Theta}}_1 + K_{vr} s_r + \tau_f \quad (3.9)$$

With $\tau_f = \dfrac{(1+k)s_r}{\|s_r\|^2 + \varepsilon}(s_f^T Z_2 \dot{\hat{\Theta}}_2 + s_f^T K_{vf} s_f)$ (2.56) and k solution of $\dot{k} = \dfrac{1}{k}\left(\dfrac{k(\|s_r\|^2 - \varepsilon)}{\|s_r\|^2 + \varepsilon}\right)(s_f^T \tau_a)$

(2.56)

And the following adaptation algorithm.

$$\dot{\hat{\Theta}}_1 = K_1 Z_1 s_r \qquad (3.10)$$

$$\dot{\hat{\Theta}}_2 = K_2 Z_2 s_f \qquad (3.11)$$

Will stabilized the Flexible link robot system. The architecture of this controller is given in Figure [3.3]

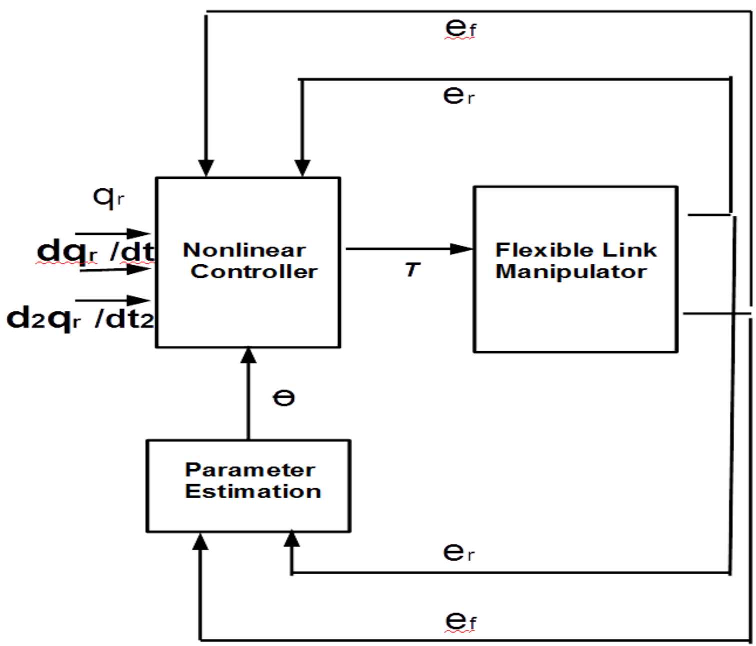

Figure 3.3 An Adaptive Control Architecture.

4.0 Conclusion

In this tutorial an overview of the control of flexible link manipulator is presented. Modeling of robots that exhibit link flexibilities are presented using assumed mode method and finite element methods. $H\infty$ robust control strategies, Nonlinear as well as adaptive control strategies were also presented.

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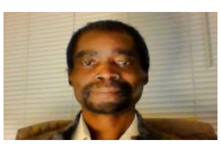

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